

# On metric characterizations of the Radon-Nikodým and related properties of Banach spaces

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**Abstract** We find a class of metric structures which do not admit bilipschitz embeddings into Banach spaces with the Radon-Nikodým property. Our proof relies on Chatterji's (1968) martingale characterization of the RNP and does not use the Cheeger's (1999) metric differentiation theory. The class includes the infinite diamond and both Laakso (2000) spaces. We also show that for each of these structures there is a non-RNP Banach space which does not admit its bilipschitz embedding.

We prove that a dual Banach space does not have the RNP if and only if it admits a bilipschitz embedding of the infinite diamond.

The paper also contains related characterizations of reflexivity and the infinite tree property.

**Keywords:** Banach space, diamond graph, geodesic, infinite tree property, Laakso space, martingale, Radon-Nikodým property, reflexivity

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# 1 Introduction and some general problems

In the recent work on metric embeddings a substantial role is played by existence and non-existence of bilipschitz embeddings of metric spaces into Banach spaces with the Radon-Nikodým property (RNP, for short), see [CK06, CK09, LN06]. At the seminar “Nonlinear geometry of Banach spaces” (Texas A & M University, August 2009) Johnson suggested the problem of metric characterization of reflexivity and the Radon-Nikodým property [Tex09, Problem 1.1].

The problem of metric characterization of the Radon-Nikodým property and reflexivity can be understood and approached in several different ways. The purpose of the present paper is to develop one of the possible approaches to it. Our approach is similar to the approach of metric characterization of superreflexivity suggested by Bourgain [Bou86] in the first paper on metric characterizations of classes of Banach spaces. This approach (for different classes of spaces) was later followed in [Bau07, BMW86, JS09, MN08, Ost11, Ost13a, Ost13+, Pis86] (see also accounts in [Pis11] and [Ost13b]). The mentioned approach is based on the notion of test spaces.

**Definition 1.1.** Let  $\mathcal{P}$  be a class of Banach spaces and let  $T = \{T_\alpha\}_{\alpha \in A}$  be a set of metric spaces. We say that  $T$  is a set of *test-spaces* for  $\mathcal{P}$  if the following two conditions are equivalent:

1.  $X \notin \mathcal{P}$ .
2. The spaces  $\{T_\alpha\}_{\alpha \in A}$  admit bilipschitz embeddings into  $X$  with uniformly bounded distortion.

*Remark 1.2.* We write  $X \notin \mathcal{P}$  rather than  $X \in \mathcal{P}$  for terminological reasons: we would like to use terms “test-spaces for reflexivity, superreflexivity, etc.” rather than “test-spaces for **non**reflexivity, **nonsuper**reflexivity, etc.”

We will be mostly interested in the following special case of Definition 1.1:

**Definition 1.3.** We say that a metric space  $X$  is a *test space* for  $\mathcal{P}$  if the bilipschitz embeddability of  $X$  into a Banach space  $Y$  is equivalent to  $Y \notin \mathcal{P}$ .

The following problems are open.

**Problem 1.4.** *Does there exist a test space for the RNP?*

**Problem 1.5.** *Does there exist a test space for reflexivity?*

*Remark 1.6.* It should be mentioned that, as we know from the well-known example of Ribe [Rib84] (see also [BL00, Theorem 10.28]), the RNP and reflexivity are not preserved by uniform homeomorphisms and therefore their metric characterizations are not included into the Ribe program. See [Bal12] and [Nao12] for description of the Ribe program.

*Remark 1.7.* The example of Ribe [Rib84] mentioned above, combined with the classical observation of [CK63] (see also [BL00, Proposition 1.11] and [Ost13b, Lemma 9.7]) that uniformly continuous maps between Banach spaces are Lipschitz for “large” distances, implies that RNP and reflexivity cannot be characterized by uniformly discrete test spaces.

The natural candidates for being test spaces for reflexivity and the RNP are the infinite diamond  $D_\omega$  and the first Laakso space  $L_\omega$  (we recall the definition of  $D_\omega$  below). These spaces are the natural candidates because it was shown in [JS09] (see also [Ost11] and [Ost13b, Section 9.3.2]) that their finite versions form collections of test spaces for superreflexivity. However, it turns out that these natural candidates for being test spaces for RNP are not such. More precisely if a Banach space  $X$  admits a bilipschitz embedding of the infinite diamond or the first Laakso space, then  $X \notin \text{RNP}$ , but there are non-RNP Banach spaces which do not admit bilipschitz embedding of these spaces. In the case of the infinite diamond these statements were proved in [Ost11]. In the case of the first Laakso space, only the first statement was proved in [Ost11]. One of the purposes of this paper is to generalize these results. In Theorems 4.1 and 4.8 we find a wide class  $\mathcal{R}$  of metric spaces containing, in addition to  $D_\omega$  and  $L_\omega$ , also the second Laakso space  $X_\omega$  (defined in Example 4.2), and such that the bilipschitz embeddability of a metric space  $M \in \mathcal{R}$  into a Banach space  $X$  implies that  $X \notin \text{RNP}$ . On the other hand, we prove (Theorem 4.9) that for each  $M \in \mathcal{R}$  there exists a non-RNP space  $X$  which does not admit bilipschitz embeddings of  $M$ .

Here we would like to mention that Cheeger and Kleiner [CK09, Corollary 1.9] used the theory of differentiability of functions on metric spaces developed by Cheeger [Che99] (see also [Kei04, KM11]) in order to show that the Laakso spaces  $L_\omega$  and  $X_\omega$  (introduced in [Laa00], see [LP01, p. 290] and [CK13, Example 1.2 and Example 1.4] for their elegant description) do not admit bilipschitz embeddings into a Banach space with the RNP. Theorem 4.1 provides a different proof of non-embeddability of the Laakso spaces  $L_\omega$  and  $X_\omega$  into Banach spaces with the RNP. See Section 4.2 for the proof for  $X_\omega$ , the  $L_\omega$  case was already considered in [Ost11]. This proof does not use the theory of differentiability of functions on metric spaces developed by Cheeger [Che99].

Now we recall the definition of infinite diamond. The *diamond graph* of level 0 is denoted  $D_0$ . It has two vertices joined by an edge of length 1.  $D_n$  is obtained from  $D_{n-1}$  as follows. Each edge of  $D_{n-1}$  is of length  $2^{-(n-1)}$ . Given an edge  $uv \in E(D_{n-1})$ , it is replaced by a quadrilateral  $u, a, v, b$  with edge lengths  $2^{-n}$ . We endow  $D_n$  with their shortest path metrics. We consider the vertex of  $D_n$  as a

subset of the vertex set of  $D_{n+1}$ , it is easy to check that this defines an isometric embedding. We introduce  $D_\omega$  as the union of the vertex sets of  $\{D_n\}_{n=0}^\infty$ . For  $u, v \in D_\omega$  we introduce  $d_{D_\omega}(u, v)$  as  $d_{D_n}(u, v)$  where  $n \in \mathbb{N}$  is any integer for which  $u, v \in V(D_n)$ . Since the natural embeddings  $D_n \rightarrow D_{n+1}$  are isometric,  $d_{D_n}(u, v)$  does not depend on the choice of  $n$  for which  $u, v \in V(D_n)$ . To the best of my knowledge the first paper in which diamond graphs  $\{D_n\}_{n=0}^\infty$  were used in Metric Geometry is [GNRS04] (conference version was published in 1999).

**Definition 1.8.** Let  $\delta > 0$ . A sequence  $\{x_i\}_{i=1}^\infty$  in a Banach space  $X$  is called a  $\delta$ -tree if  $x_i = \frac{1}{2}(x_{2i} + x_{2i+1})$  and  $\|x_{2i} - x_i\| = \|x_{2i+1} - x_i\| \geq \delta$ . We say that a Banach space  $X$  has the *infinite tree property* if it contains a bounded  $\delta$ -tree for some  $\delta > 0$ .

**Theorem 1.9** ([Ost11]). *A bilipschitz embeddability of  $D_\omega$  into a Banach space  $Y$  implies the infinite tree property of  $Y$ .*

The mentioned above results about  $D_\omega$  can be obtained by combining Theorem 1.9 with known results on the RNP. Namely, it is known [BL00, page 111] that Banach spaces with the infinite tree property do not have the RNP. On the other hand, Bourgain and Rosenthal [BR80] (see also [BL00, Example 5.30]) constructed an example of a Banach space without the RNP which does not have the infinite tree property.

In view of Theorem 1.9 the following result which we prove in this paper could be considered as a strengthening of a result of Stegall [Ste75], who proved that dual Banach spaces without the RNP have the infinite tree property.

**Theorem 1.10.** *A dual Banach space does not have the RNP if and only if it admits a bilipschitz embedding of  $D_\omega$ .*

This strengthening is not immediate because at the moment it is not known whether the infinite tree property of a Banach space  $Y$  implies the bilipschitz embeddability of  $D_\omega$  into  $Y$ . In this connection we observe that if we widen the notion of a test space to what we call a *submetric test-space*, we easily get a characterization of the infinite tree property. We mean following definition.

**Definition 1.11.** A *submetric test-space* for a class  $\mathcal{P}$  of Banach spaces is defined as a metric space  $T$  with a marked subset  $S \subset T \times T$  such that the following conditions are equivalent for a Banach space  $X$ :

1.  $X \notin \mathcal{P}$ .
2. There exist a constant  $0 < C < \infty$  and an embedding  $f : T \rightarrow X$  satisfying the condition

$$\forall (x, y) \in S \quad d_T(x, y) \leq \|f(x) - f(y)\| \leq C d_T(x, y). \quad (1)$$

An embedding satisfying (1) is called a *partially bilipschitz* embedding. Pairs  $(x, y)$  belonging to  $S$  are called *active*.

**Theorem 1.12.** *The class of Banach spaces with the infinite tree property admits a submetric characterization in terms of the metric space  $D_\omega$  with the set of active pairs defined as follows: a pair is active if and only if it is a pair of vertices of a quadrilateral introduced in one of the steps.*

In this paper we find a submetric test space for reflexivity (Theorem 5.1). As we have mentioned above (Problem 1.5) the problem of existence of a *metric* test space for reflexivity remains open.

## 2 Dual non-RNP spaces, proof of Theorem 1.10

*Proof.* It is well-known (see [Ste75, Theorems A and 2]) that a dual Banach space  $X^*$  does not have the RNP if and only if  $X$  contains a separable subspace  $Y$  such that  $Y^*$  is nonseparable. First we prove this result in the case where  $X$  is separable. We use the construction of Stegall [Ste75, Theorem 1] (see also [DU77, pp. 192–195]). In the case where  $X$  is a separable Banach space and  $X^*$  is a nonseparable Banach space, and  $\varepsilon > 0$ , he constructed

- A collection  $\{x_{n,i}\}_{n=0, i=0}^{2^n-1}$  of vectors in  $X$  satisfying  $\|x_{n,i}\| < 1 + \varepsilon$ .
- A collection  $\{W_{n,i}\}_{n=0, i=0}^{2^n-1}$  of nonempty weak\* compact convex subsets in  $B_{X^*}$  (the unit ball of  $X^*$ ) such that

$$W_{n,i} \supset W_{n+1,2i} \cup W_{n+1,2i+1} \quad (2)$$

$$\{W_{n,i}\}_{i=0}^{2^n-1} \text{ are pairwise disjoint.} \quad (3)$$

These collections are such that if by  $\Delta$  we denote the set

$$\bigcap_{n=0}^{\infty} \left( \bigcup_{i=0}^{2^n-1} W_{n,i} \right),$$

by  $C(\Delta)$  we denote the space of weak\* continuous functions on  $\Delta$ , by  $h_{n,i}$  denote the indicator function of  $W_{n,i} \cap \Delta$ , and by  $T : X \rightarrow C(\Delta)$  denote the natural embedding, the following condition holds:

$$\sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \|Tx_{n,i} - h_{n,i}\| < \varepsilon. \quad (4)$$

Observe that the function  $h_{n,i}$  is continuous on  $\Delta$  because of the conditions (2) and (3).

First we use these collections to construct a bounded  $\delta$ -tree in  $X^*$ . The existence of such tree is well known, see [BL00, p. 114]. We present the details because some of the specific properties of our construction are needed for the embedding of  $D_\omega$ .

We pick a vector (arbitrarily) in each of the sets  $\{\Delta \cap W_{n,i}\}_{i=0}^{2^n-1}$ . To conform with our notation for trees we denote the vector picked in  $\Delta \cap W_{n,i}$  by  $y_j^n$ , where  $j = i + 2^n$ , so  $j = 2^n, \dots, 2^{n+1} - 1$ . We define  $y_j^n$  with  $j = 2^{n-1}, \dots, 2^n - 1$  by

$$y_j^n = \frac{1}{2}(y_{2j}^n + y_{2j+1}^n). \quad (5)$$

Next we define  $y_j^n$  with  $j = 2^{n-2}, \dots, 2^{n-1} - 1$  using (5). We continue in an obvious way and define all  $\{y_j^n\}$  for  $j = 1, \dots, 2^{n+1} - 1$ .

Our next purpose is to show that for each  $j = 1, \dots, 2^n - 1$  the condition

$$|(y_{2j}^n - y_{2j+1}^n)(x_{k+1,2r})| > 1 - 2\varepsilon \quad (6)$$

holds, where  $k$  and  $r$  are determined by the condition:  $j = 2^k + r$  with  $r < 2^k$ . To show this we observe, by using (5), that  $y_{2j}^n$  is a convex combination of

$$y_m^n \quad \text{for } m = 2^n + 2^{n-k}r, 2^n + 2^{n-k}r + 1, \dots, 2^n + 2^{n-k}r + 2^{n-k-1} - 1. \quad (7)$$

On the other hand,  $y_{2j+1}^n$  for  $j = 2^k + r$  with  $r < 2^k$  is a convex combination of

$$y_m^n \quad \text{for } m = 2^n + 2^{n-k}r + 2^{n-k-1}, 2^n + 2^{n-k}r + 2^{n-k-1} + 1, \dots, 2^n + 2^{n-k}r + 2^{n-k} - 1. \quad (8)$$

Observe that elements of (7) are contained in  $\Delta \cap W_{k+1,2r}$  and elements of (8) are contained in  $\Delta \cap W_{k+1,2r+1}$ . Therefore  $h_{k+1,2r}$  satisfies  $h_{k+1,2r}(z) = 1$  for each  $z$  in (7) and  $h_{k+1,2r}(z) = 0$  for each  $z$  in (8). Applying (4) we get

$$z(x_{k+1,2r}) > 1 - \varepsilon \quad (9)$$

for each  $z$  in (7) and

$$|z(x_{k+1,2r})| < \varepsilon \quad (10)$$

for each  $z$  in (8). It is clear that (9) and (10) continue to hold if  $z$  is a convex combination of vectors in (7) and (8), respectively. Inequality (6) follows.

We construct such finite sequences  $\{y_j^n\}_{j=1}^{2^{n+1}-1}$  for each  $n = 0, 1, 2, \dots$ . We introduce  $\{y_j\}_{j=1}^\infty$  by  $y_j = w^* - \lim_{n \rightarrow \infty} y_j^n$ . It is clear that (6) implies

$$|(y_{2j} - y_{2j+1})(x_{k+1,2r})| \geq 1 - 2\varepsilon. \quad (11)$$

Therefore  $\|y_{2j} - y_{2j+1}\| \geq \frac{1-2\varepsilon}{1+\varepsilon}$  and the sequence  $\{y_j\}_{j=1}^\infty$  forms a  $\delta$ -tree with  $\delta = \frac{1-2\varepsilon}{2(1+\varepsilon)}$ . It is also clear that this tree is contained in the unit ball of  $X^*$ .

Observe that using (4) in the case where  $n = i = 0$  we get  $z(x_{0,0}) \geq 1 - \varepsilon$  for each  $z \in \Delta$ . Thus  $y_j(x_{0,0}) \geq 1 - \varepsilon$  and

$$\|y_j\| \geq \frac{1-\varepsilon}{1+\varepsilon} \quad \text{for each } j \in \mathbb{N}. \quad (12)$$

We need to derive one more consequence of inequalities (9) and (10) and our construction. We define the *tail* of  $y_t$  in the tree  $\{y_j\}_{j=1}^\infty$  as the set

$$\{y_t, y_{2t}, y_{2t+1}, y_{4t}, y_{4t+1}, y_{4t+2}, y_{4t+3}, \dots, y_{2^k t+1}, \dots, y_{2^k t+2^k-1}, \dots\}.$$

(This set can be informally described as the set of all “branches” which “grow” out of  $y_t$ .) We are going to use the observation that (9) and (10) imply that

$$|y_m(x_{k+1,2r})| \geq 1 - \varepsilon \quad (13)$$

for all  $y_m$  in the tail of  $y_{2j}$  and

$$|y_m(x_{k+1,2r})| \leq \varepsilon \quad (14)$$

for all  $y_m$  in the tail of  $y_{2j+1}$ .

Now we construct a bilipschitz embedding of  $D_\omega$  into  $X^*$ . Vertices of  $D_0$  are mapped in the following way: one vertex is mapped onto  $0 \in X^*$  and the other onto  $y_1$  (the first element of the tree). We continue in the following way: two new vertices of  $D_1$  are mapped onto  $\frac{y_2}{2}$  and  $\frac{y_3}{2}$ , respectively. The result will be a bilipschitz image of  $D_1$  because by (11), we have  $\frac{1-2\varepsilon}{1+\varepsilon} \leq \|y_2 - y_3\| \leq \|y_2\| + \|y_3\| \leq 2$ . We proceed in an obvious way. To make this more clear we describe the next step.

In the obtained image of  $D_1$  two edges correspond to  $\frac{y_2}{2}$  and two edges correspond to  $\frac{y_3}{2}$ . Here and below we say that an edge *corresponds to a vector*  $z$  if the difference between the images of the ends of the edge is  $\pm z$ . In the diamond graphs edges are replaced by quadrilaterals. In the images we replace the corresponding vectors by parallelograms. When we say that a vector  $z$  corresponding to an edge  $uv$  is *replaced by a parallelogram with sides*  $x$  *and*  $y$ , where  $x + y = z$ , we mean the following. If  $z = f(v) - f(u)$  and  $u, a, v, b$  is the quadrilateral which replaces the edge  $uv$ , then  $f(a) = f(u) + x$  and  $f(b) = f(u) + y$ , so for the obtained mapping the edges  $ua$  and  $bv$  correspond to  $x$  and the edges  $ub$  and  $av$  correspond to  $y$ .

Now we return to the construction of the embedding of  $D_\omega$ . The vector  $\frac{y_2}{2}$  is replaced by a parallelogram with sides  $\frac{y_4}{4}$  and  $\frac{y_5}{4}$ , and the mapping  $f$  is extended in the described in the previous paragraph way to each quadrilateral which replaces edges corresponding to  $\frac{y_2}{2}$ . In the next step the edge corresponding to  $\frac{y_3}{2}$  is replaced by a parallelogram with sides  $\frac{y_6}{4}$  and  $\frac{y_7}{4}$ , and the mapping  $f$  is extended in the described in the previous paragraph way to each quadrilateral which replaces edges corresponding to  $\frac{y_3}{2}$ . So on.

Since  $\|y_j\| \leq 1$  for each  $j \in \mathbb{N}$ , the constructed in such a way embedding  $f$  of  $D_\omega$  into  $X^*$  is 1-Lipschitz. It remains to show that it is bilipschitz. Here we use some notions introduced by Johnson and Schechtman [JS09] in their study of embeddings of finite diamonds (see also [Ost13b, Section 9.3.2]).

Namely, for any edge  $uv$  in  $D_k$  we let  $S(u, v)$  be the union of the following sequence of sets of vertices:

1. Vertices of the quadrilateral which replaces the edge  $uv$
2. Vertices of the quadrilaterals which replace the edges of the quadrilateral from the previous item.



3. Vertices of the quadrilaterals which replace the edges of the quadrilateral from the previous item.
4. So on.

We call the set  $D(u, v)$  a *subdiamond* of  $D_\omega$ . It is easy to see that for any two vertices  $w, z \in D_\omega$  there is a well-defined notion of the *smallest subdiamond* containing them. Let  $D(u, v)$  be the smallest subdiamond containing  $w$  and  $z$ . Let  $u, a, v, b$  be the quadrilateral which replaces the edge  $uv$  (when we form the next diamond). Then

$$D(u, v) = (D(u, a) \cup D(a, v)) \bigcup (D(u, b) \cup D(b, v)).$$

We call the sets  $(D(u, a) \cup D(a, v))$  and  $(D(u, b) \cup D(b, v))$  the *a-side* of the subdiamond  $D(u, v)$  and the *b-side* of the subdiamond  $D(u, v)$ , respectively. There are two possibilities:

- $w$  and  $z$  are on the same side of  $D(u, v)$ .
- $w$  and  $z$  are on different sides of  $D(u, v)$ .

It is clear that in the first case we may assume that both  $w$  and  $z$  are on the *a-side* of  $D(u, v)$ . Also since  $D(u, v)$  is the *smallest* subdiamond containing  $w$  and  $z$ ,  $w$  and  $z$  cannot be both in  $D(u, a)$  or both in  $D(a, v)$ . So we may assume that  $w \in D(u, a)$  and  $z \in D(a, v)$ .

We have  $f(a) - f(u) = \frac{y_{2j}}{2^k}$  and  $f(v) - f(a) = \frac{y_{2j+1}}{2^k}$  for some  $j \in \mathbb{N}$ , where  $k$  is determined by  $2j = 2^k + r$  with  $0 \leq r < 2^k$ . (It can be that  $f(a) - f(u) = \frac{y_{2j+1}}{2^k}$  and  $f(v) - f(a) = \frac{y_{2j}}{2^k}$ , but for our argument this does not matter. However, to deal with the alternative case we would need analogues of (13) and (14)  $x_{k+1, 2r+1}$ .)

One can prove by induction the following two statements.

- (a) The difference  $f(a) - f(w)$  is a linear combination with nonnegative coefficients of vectors contained in the tail of  $y_{2j}$  and the sum  $\sigma_{a,w}$  of the coefficients of this linear combination is equal to  $d_{D_\omega}(a, w)$ .
- (b) The difference  $f(z) - f(a)$  is a linear combination with nonnegative coefficients of vectors contained in the tail of  $y_{2j+1}$  and the sum  $\sigma_{z,a}$  of the coefficients of this linear combination is equal to  $d_{D_\omega}(z, a)$ .

To estimate  $\|f(z) - f(w)\|$  from below we need to consider two cases:  $d_{D_\omega}(z, a) \leq d_{D_\omega}(a, w)$  and  $d_{D_\omega}(z, a) \geq d_{D_\omega}(a, w)$ . We consider the first case only, the second case is similar. We have

$$\begin{aligned} \|f(z) - f(w)\| &\geq \left| \frac{(f(z) - f(a))(x_{k+1,r}) + (f(a) - f(w))(x_{k+1,r})}{1 + \varepsilon} \right| \\ &\stackrel{\text{(a)\&(b)\&(13)\&(14)}}{\geq} \frac{d_{D_\omega}(a, w)(1 - \varepsilon) - d_{D_\omega}(z, a)\varepsilon}{1 + \varepsilon} \\ &\geq \frac{d_{D_\omega}(a, w)(1 - 2\varepsilon)}{1 + \varepsilon} \geq \frac{d_{D_\omega}(w, z)(1 - 2\varepsilon)}{2(1 + \varepsilon)}. \end{aligned}$$



Now we consider the **different sides** case. In this case there are two subcases:

- (A) Either we have both  $d_{D_\omega}(w, u) \leq d_{D_\omega}(w, v)$  and  $d_{D_\omega}(z, u) \leq d_{D_\omega}(z, v)$ , or both  $d_{D_\omega}(w, u) \geq d_{D_\omega}(w, v)$  and  $d_{D_\omega}(z, u) \geq d_{D_\omega}(z, v)$ .
- (B) One of the vertices  $w$  and  $z$  closer to  $u$  and the other is closer to  $v$ . We may assume  $d_{D_\omega}(w, u) < d_{D_\omega}(w, v)$  and  $d_{D_\omega}(z, u) > d_{D_\omega}(z, v)$ .

In the subcase (A) we use almost the same argument as above. We may assume that both  $w$  and  $z$  are at least as close to  $u$  as to  $v$  and that  $d_{D_\omega}(w, u) \geq d_{D_\omega}(z, u)$ . We have  $f(a) - f(u) = \frac{y_{2j}}{2^k}$  and  $f(b) - f(u) = \frac{y_{2j+1}}{2^k}$  for some  $j \in \mathbb{N}$ , where  $k$  is determined by  $2j = 2^k + r$  with  $0 \leq r < 2^k$ . (It can happen that  $f(a) - f(u) = \frac{y_{2j+1}}{2^k}$  and  $f(b) - f(u) = \frac{y_{2j}}{2^k}$ , but this case can be treated similarly.) We also may assume that  $w$  is on the  $a$ -side.

One can prove by induction the following two statements.

- (c) The difference  $f(w) - f(u)$  is a linear combination with nonnegative coefficients of vectors contained in the tail of  $y_{2j}$  and the sum  $\sigma_{w,u}$  of the coefficients of this linear combination is equal to  $d_{D_\omega}(w, u)$ .
- (d) The difference  $f(z) - f(u)$  is a linear combination with nonnegative coefficients of vectors contained in the tail of  $y_{2j+1}$  and the sum  $\sigma_{z,u}$  of the coefficients of this linear combination is equal to  $d_{D_\omega}(z, u)$ .

We have

$$\begin{aligned}
\|f(w) - f(z)\| &\geq \left| \frac{(f(w) - f(u))(x_{k+1,r}) + (f(u) - f(z))(x_{k+1,r})}{1 + \varepsilon} \right| \\
&\stackrel{(c) \& (d) \& (13) \& (14)}{\geq} \frac{d_{D_\omega}(w, u)(1 - \varepsilon) - d_{D_\omega}(z, u)\varepsilon}{1 + \varepsilon} \\
&\geq \frac{d_{D_\omega}(w, u)(1 - 2\varepsilon)}{1 + \varepsilon} \geq \frac{d_{D_\omega}(w, z)(1 - 2\varepsilon)}{2(1 + \varepsilon)}.
\end{aligned}$$

Now we consider subcase (B). We may assume that  $w$  is contained in the subdiamond  $D(u, a)$  and  $z$  is contained in the subdiamond  $D(b, v)$ , where  $u, a, v, b$  is the quadrilateral which replaces the edge  $uv$ . We also may assume that  $f(a) - f(u) = \frac{y_{2j+1}}{2^k}$ ,  $f(v) - f(b) = \frac{y_{2j+1}}{2^k}$ , and  $f(b) - f(u) = \frac{y_{2j}}{2^k}$  for some  $j \in \mathbb{N}$ , where  $k$  is determined by  $2j = 2^k + r$  with  $0 \leq r < 2^k$ . One can prove by induction the following statement.

- (e) The differences  $f(w) - f(u)$  and  $f(z) - f(b)$  are linear combinations with nonnegative coefficients of vectors contained in the tail of  $y_{2j+1}$  and the sums  $\sigma_{w,u}$  and  $\sigma_{z,b}$  of the coefficients of these linear combinations are equal to  $d_{D_\omega}(w, u)$  and  $d_{D_\omega}(z, u)$ , respectively.

Observe that in the considered case the diameter of  $D(u, v)$  is equal to  $\frac{1}{2^{k-1}}$ , the diameters of  $D(u, a)$  and  $D(b, v)$  are equal to  $\frac{1}{2^k}$ . On the other hand, we have

$$\begin{aligned}
& \|f(z) - f(w)\| \\
& \geq \left| \frac{(f(b) - f(u))(x_{k+1,r}) + (f(z) - f(b))(x_{k+1,r}) + (f(u) - f(w))(x_{k+1,r})}{1 + \varepsilon} \right| \\
& \stackrel{(e) \& (13) \& (14)}{\geq} \frac{\frac{1}{2^k}(1 - \varepsilon) - d_{D_\omega}(z, b)\varepsilon - d_{D_\omega}(u, w)\varepsilon}{1 + \varepsilon} \\
& \geq \frac{1}{2^k} \cdot \frac{1 - 3\varepsilon}{1 + \varepsilon} \\
& \geq \frac{d_{D_\omega}(w, z)(1 - 3\varepsilon)}{2(1 + \varepsilon)}.
\end{aligned}$$

This completes the proof in the case where  $X^*$  is a nonseparable dual of a separable Banach space.

Now we consider the case where  $X$  is nonseparable, but contains a separable subspace  $Y$  such that  $Y^*$  is nonseparable. In this case we apply Stegall's construction mentioned at the beginning of the proof to  $Y$  and denote the obtained collection of vectors in  $Y \subset X$  by  $\{x_{n,i}\}_{n=0}^\infty_{i=0}^{2^n-1}$ , and the obtained collection of nonempty weak\* compact convex subsets in  $B_{Y^*}$  by  $\{V_{n,i}\}_{n=0}^\infty_{i=0}^{2^n-1}$ . We let  $W_{n,i}$  be the set of all norm preserving extensions of functionals of the set  $V_{n,i}$  to the whole space  $X$ . It is clear that  $W_{n,i}$  are nonempty weak\* compact convex subsets in  $B_{X^*}$ , and that they satisfy the conditions (2) and (3). It is also easy to verify that if we construct  $\Delta$  in the same way as before, the condition (4) also continues to hold. Therefore everything in the proof can be done in the same way as in the case where  $X$  is separable.  $\square$

### 3 A submetric test space for the infinite tree property, proof of Theorem 1.12

*Proof of Theorem 1.12.* On one hand, if we analyze the proof of Theorem 1.9 in [Ost11], we see that we used the bilipschitz condition only for pairs of points which are in the same quadrilateral.

On the other hand, let  $\{x_i\}_{i=1}^\infty$  be a bounded  $\delta$ -tree in a Banach space  $Z$ . First we shift the tree in order to achieve the situation in which it is bounded both from below (in the sense that  $\exists c > 0 \forall i \in \mathbb{N} \|x_i\| \geq c$ ) and from above. Now we construct the image of the submetric space  $(D_\omega, S_\omega)$  in the following way (the construction of [Ost11] backward):

We map vertices of  $D_0$  onto 0 and  $x_1$ , respectively. We map the new vertices  $a$  and  $b$  of the quadrangle which replaces  $D_0$  to  $x_2/2$  and  $x_3/2$ , respectively. It is clear that because  $x_1$  is the sum of  $x_2/2$  and  $x_3/2$ , that all edges of  $D_1$  correspond to one of them (correspond in the sense that they are differences between the vectors corresponding to the vertices).

We continue as follows. Since  $x_2/2 = x_4/4 + x_5/4$  we let the edges of the second level in quadruples having  $x_2/2$  as the difference between the top and the bottom to be: (the bottom)+ $x_4/4$  and (the bottom)+ $x_5/4$ . We continue in an obvious way.

It is easy to verify that the fact that the tree is a bounded  $\delta$ -tree and norms of its elements are bounded away from zero implies that we get a partial bilipschitz embedding of the submetric space  $(D_\omega, S_\omega)$  into the Banach space  $Z$ . (To visualize the proof it is worthwhile to sketch a  $D_2$  and to label edges by differences between the vectors corresponding to their end vertices.)  $\square$

## 4 Classes of metric spaces which do not admit bilipschitz embeddings into spaces with the Radon-Nikodým property

The conditions implying non-embeddability, which we are going to present in this section are of the type: Any bilipschitz image of a metric space  $X$  in a Banach space  $Y$  contains a set which can be used to form a bounded divergent martingale (the values of the martingale are multiples of differences between images of certain points of the metric space). We use the following result of Chatterji [Cha68] (see also [BL00], [Bou83], [DU77], and [Pis11]): A Banach space  $Y$  has the RNP if and only if each bounded  $Y$ -valued martingale converges.

### 4.1 Spaces with thick families of geodesics between some pairs of points

The purpose of this section is to prove the following generalization of the result of [Ost11]:

**Theorem 4.1.** *Let  $(X, d)$  be a metric space satisfying the following two conditions:*

- (1) *There are two points  $u$  and  $v$  in  $X$  and infinitely many marked geodesics between them in the completion  $\tilde{X}$  of  $X$  such that the following condition is satisfied. (Not all of the geodesics between  $u$  and  $v$  have to be marked.)*
- (2) *For any two points  $u_0$  and  $v_0$  on a marked  $uv$ -geodesic there are points*

$$w_0 = u_0, w_1, \dots, w_{n-1}, w_n = v_0$$

*which also lie on some marked  $uv$ -geodesic, and their order on the geodesic coincides with the order in which they are listed; and there are points  $\{z_i, \tilde{z}_i\}_{i=1}^n$  such that*

- (a) *The points  $w_0 = u_0, z_1, w_1, z_2, \dots, w_{n-1}, z_n, w_n = v_0$  lie on a marked  $uv$ -geodesic and are listed in their order on the geodesic.*
- (b) *The points  $w_0 = u_0, \tilde{z}_1, w_1, \tilde{z}_2, \dots, w_{n-1}, \tilde{z}_n, w_n = v_0$  lie on a marked  $uv$ -geodesic and are listed in their order on the geodesic.*

These geodesics have to be different because of the next two conditions.

(c)  $d(w_i, z_i) = d(w_i, \tilde{z}_i)$  and  $d(w_{i-1}, z_i) = d(w_{i-1}, \tilde{z}_i)$ .

(d)

$$\sum_{i=1}^n d(z_i, \tilde{z}_i) \geq cd(u_0, v_0), \quad (15)$$

where  $c$  depends on  $X$  but not on the choice of  $u_0$  and  $v_0$ .

Then the metric space  $(X, d)$  does not admit a bilipschitz embedding into a Banach space with the Radon-Nikodým property.

*Proof.* We assume that  $(X, d)$  admits a bilipschitz embedding  $f : X \rightarrow Y$  into a Banach space  $Y$  and show that there exists a bounded divergent martingale  $\{M_i\}_{i=0}^\infty$  on  $(0, 1]$  with values in  $Y$ . We assume that

$$\ell d(x, y) \leq \|f(x) - f(y)\|_Y \leq d(x, y) \quad (16)$$

for some  $\ell > 0$ . We assume that  $d(u, v) = 1$  (dividing all distances in  $X$  by  $d(u, v)$ , if necessary).

Each function in the martingale  $\{M_i\}_{i=0}^\infty$  will be obtained in the following way. We consider some finite sequence  $V = \{v_i\}_{i=0}^m$  of points on a (not necessarily marked)  $uv$ -geodesic, satisfying  $v_0 = u$ ,  $v_m = v$  and  $d(u, v_{k+1}) \geq d(u, v_k)$ . We define  $M_V$  as the function on  $(0, 1]$  whose value on the interval  $(d(u, v_k), d(u, v_{k+1})]$  is equal to

$$\frac{f(v_{k+1}) - f(v_k)}{d(v_k, v_{k+1})}.$$

It is clear that (16) implies that  $\|M_V(t)\| \leq 1$  for any collection  $V$  and any  $t \in (0, 1]$ . Also it is clear that an infinite collection of such functions  $\{M_{V(k)}\}_{k=0}^\infty$  forms a martingale if for each  $k \in \mathbb{N}$  the sequence  $V(k)$  contains  $V(k-1)$  as a subsequence. So it remains to find such increasing collection of sequences  $\{V(k)\}_{k=0}^\infty$  for which the martingale  $\{M_{V(k)}\}_{k=0}^\infty$  diverges. We denote  $M_{V(k)}$  by  $M_k$ .

We let  $V(0) = \{u, v\}$  and so  $M_0$  is a constant function on  $(0, 1]$  taking value  $f(v) - f(u)$ . In the next step we apply the condition **(2)** to  $v_0 = v$  and  $u_0 = u$  and find the corresponding sequences  $\{w_i\}_{i=0}^n$  and  $\{z_i, \tilde{z}_i\}_{i=1}^n$ . We let  $V(1) = \{w_i\}_{i=0}^n$ . Observe, that in this step we cannot claim any nontrivial estimates for  $\|M_1 - M_0\|_{L_1(Y)}$  from below because we have not made any nontrivial assumptions on this step of the construction. Lower estimates for martingale differences in our argument are obtained only for differences of the form  $\|M_{2k} - M_{2k-1}\|_{L_1(Y)}$ .

We choose  $V(2)$  to be of the form

$$w_0, z'_1, w_1, z'_2, w_n, \dots, z'_n, w_n, \quad (17)$$

where each  $z'_i$  is either  $z_i$  or  $\tilde{z}_i$  depending on the behavior of the mapping  $f$ . We describe this dependence below. Observe that by condition (c) of Theorem 4.1, the corresponding partition of the interval  $(0, 1]$  does not depend on whether we choose  $z_i$  or  $\tilde{z}_i$ .

To make the choice of  $z'_i$  we consider the quadrilateral  $w_{i-1}, z_i, w_i, \tilde{z}_i$ . Inequality (16) implies  $\|f(z_i) - f(\tilde{z}_i)\| \geq \ell d(z_i, \tilde{z}_i)$ . Consider two pairs of vectors corresponding to two different choices of  $z'_i$ :

**Pair 1:**  $f(w_i) - f(z_i), f(z_i) - f(w_{i-1})$ . **Pair 2:**  $f(w_i) - f(\tilde{z}_i), f(\tilde{z}_i) - f(w_{i-1})$ .

The inequality  $\|f(z_i) - f(\tilde{z}_i)\| \geq \ell d(z_i, \tilde{z}_i)$  implies that at least one of the following is true

$$\left\| \frac{f(w_i) - f(z_i)}{d(w_i, z_i)} - \frac{f(z_i) - f(w_{i-1})}{d(z_i, w_{i-1})} \right\| \geq \frac{\ell}{2} d(z_i, \tilde{z}_i) \left( \frac{1}{d(w_i, z_i)} + \frac{1}{d(z_i, w_{i-1})} \right) \quad (18)$$

or

$$\left\| \frac{f(w_i) - f(\tilde{z}_i)}{d(w_i, \tilde{z}_i)} - \frac{f(\tilde{z}_i) - f(w_{i-1})}{d(\tilde{z}_i, w_{i-1})} \right\| \geq \frac{\ell}{2} d(z_i, \tilde{z}_i) \left( \frac{1}{d(w_i, \tilde{z}_i)} + \frac{1}{d(\tilde{z}_i, w_{i-1})} \right) \quad (19)$$

We pick  $z'_i$  to be  $z_i$  if the left-hand side of (18) is larger than the left-hand side of (19), and pick  $z'_i = \tilde{z}_i$  otherwise.

Let us estimate  $\|M_2 - M_1\|_1$ . First we estimate the part of this difference corresponding to the interval  $(d(w_0, w_{i-1}), d(w_0, w_i)]$ . Since the restriction of  $M_2$  to the interval  $(d(w_0, w_{i-1}), d(w_0, w_i)]$  is a two-valued function, and  $M_1$  is constant on the interval, the integral

$$\int_{d(w_0, w_{i-1})}^{d(w_0, w_i)} \|M_2 - M_1\| dt \quad (20)$$

can be estimated from below in the following way. Denote the value of  $M_2$  on the first part of the interval by  $x$ , the value on the second by  $y$ , the value of  $M_1$  on the whole interval by  $z$ , the length of the first interval by  $A$  and of the second by  $B$ . We have: the desired integral is equal to  $A\|x - z\| + B\|y - z\|$  and therefore can be estimated in the following way:

$$\begin{aligned} A\|x - z\| + B\|y - z\| &\geq \max\{\|x - z\|, \|y - z\|\} \cdot \min\{A, B\} \\ &\geq \frac{1}{2}\|x - y\| \min\{A, B\}. \end{aligned}$$

Therefore, assuming without loss of generality that the left-hand side of (18) is larger than the left-hand side of (19), the integral in (20) can be estimated from below by

$$\begin{aligned} &\frac{1}{2} \left\| \frac{f(w_i) - f(z_i)}{d(w_i, z_i)} - \frac{f(z_i) - f(w_{i-1})}{d(z_i, w_{i-1})} \right\| \cdot \min\{d(w_i, z_i), d(z_i, w_{i-1})\} \\ &\geq \frac{1}{4} \ell d(z_i, \tilde{z}_i) \left( \frac{1}{d(w_i, z_i)} + \frac{1}{d(z_i, w_{i-1})} \right) \cdot \min\{d(w_i, z_i), d(z_i, w_{i-1})\} \\ &\geq \frac{1}{4} \ell d(z_i, \tilde{z}_i). \end{aligned}$$

Summing over all intervals and using the condition (15), we get  $\|M_2 - M_1\| \geq \frac{1}{4} \ell c d(u, v)$ .

Now we apply the condition **(2)** for each pair of consecutive points in the sequence

$$w_0, z'_1, w_1, z'_2, w_2, \dots, z'_n, w_n, \quad (21)$$

where each  $z'_i$  is either  $z_i$  or  $\tilde{z}_i$  depending on the choice made above. We list all of the obtained  $w$ -points in one list

$$w_0^1, w_1^1, w_2^1, \dots, w_{n(1)}^1, \quad (22)$$

and the obtained  $z$ -points as two collections:

$$z_1^1, z_2^1, \dots, z_{n(1)}^1,$$

and

$$\tilde{z}_1^1, \tilde{z}_2^1, \dots, \tilde{z}_{n(1)}^1.$$

Observe that the whole sequence (22) does not have to be on the same marked geodesic, only pieces which correspond to pairs of consecutive points in the list (21) are on marked geodesics. However this implies that any list of the form

$$w_0^1, z_1^1, w_1^1, z_2^1, w_2^1, \dots, z_{n(1)}^1, w_{n(1)}^1,$$

where each  $z_1^1$  is either  $z_1^1$  or  $\tilde{z}_1^1$  is on some (not necessarily marked)  $uv$ -geodesic. Because of this we can proceed in the same way as before in our construction of the  $Y$ -valued functions  $M_3$  and  $M_4$ . Again, we have no estimate for  $\|M_2 - M_3\|$ , but we get the same estimate for  $\|M_3 - M_4\|$ . We proceed in an obvious way. As a result we get a bounded divergent martingale.  $\square$

## 4.2 Application to the second Laakso space

Our next goal is to show that the second Laakso space [Laa00], whose construction we present following Cheeger and Kleiner [CK13, Example 1.4] satisfies the conditions of Theorem 4.1, and thus to get a different proof of the result of Cheeger and Kleiner [CK09, Corollary 1.7] stating that this space does not admit bilipschitz embeddings into Banach spaces with the RNP.

**Example 4.2 (Second Laakso space).** *We construct this space as an inductive limit of graph thickenings, that is, graphs in which edges are isometric to line segments of the corresponding lengths, and elements of the edges (not only ends) are elements of the metric spaces. We start with a space  $X_0$  which has two vertices and an edge of length 1 joining them. Given  $X_{i-1}$  we construct  $X_i$  in two steps.*

**Step 1.** *We replace each edge in  $X_i$  by a path of the same length consisting of three edges of equal length. In other words we trisect each edge in the sense that we insert new vertices after each third of it. We denoted the set of all new vertices by  $N_i$  and the obtained graph (topologically the same as  $X_i$ , but with many new vertices) by  $X_i'$ .*

**Step 2.** We consider two copies of  $X'_i$  and paste them at the respective vertices of  $N_i$ . We introduce  $X_{i+1}$  as the obtained graph thickening with its shortest path distance. More formally, we let  $X_{i+1}$  be the set of equivalence classes of  $X'_i \times \{0, 1\}$  with respect to the equivalence given by  $(v, 0) \sim (v, 1)$  for all  $v \in N_i$  (all other equivalence classes are one-element sets).

We consider  $X_{i+1}$  as a metric space with its shortest path distance. Observe that there are natural isometric embeddings  $X_i \rightarrow X_{i+1}$  (we identify  $X_i$  with  $X_i \times \{0\}$ ). We let  $X_\omega$  be the union of  $X_i$ , with the metric  $d_{X_\omega}(u, v)$  defined as  $d_{X_\omega}(u, v) = d_{X_i}(u, v)$ , where  $i \in \mathbb{N}$  is large enough so that  $u, v \in X_i$ .

**Proposition 4.3.** *The space  $X_\omega$  satisfies all conditions of Theorem 4.1 if we pick  $u$  and  $v$  to be the vertices of  $X_0$  and consider all geodesics joining them in any of  $X_i$  as marked.*

*Proof.* Let  $u_0$  and  $v_0$  be two points on a (marked)  $uv$ -geodesic. Let  $j \in \mathbb{N}$  be such that  $u_0, v_0 \in X_j$ . We trisect all edges of  $X_j$  and let  $a_0^1, \dots, a_{m(1)}^1$  be the vertices of  $N_j$  on one of the geodesics  $S$  joining  $u_0$  and  $v_0$ , in the order, in which we meet them travelling from  $u_0$  to  $v_0$ . We denote  $d_{X_\omega}$  by  $d$ . If  $d(a_0^1, a_{m(1)}^1) \geq \frac{1}{2}d(u_0, v_0)$  (the number  $\frac{1}{2}$  can be replaced by any other number in  $(0, 1)$ , this will affect only the constant  $c$  in (15)), we let  $n = m(1)$  and the vertices  $w_0, w_1, \dots, w_n$  (see condition (2) in Theorem 4.1) be given by  $w_0 = u_0, w_1 = a_1^1, \dots, w_{n-1} = a_{n-1}^1, w_n = v_0$ .

If  $d(a_0^1, a_{m(1)}^1) < \frac{1}{2}d(u_0, v_0)$ , we repeat the trisection (now we trisect edges of  $X_{j+1}$ ) and let  $a_0^2, \dots, a_{m(2)}^2$  be vertices of the new trisection on  $S$ . If  $d(a_0^2, a_{m(2)}^2) \geq \frac{1}{2}d(u_0, v_0)$ , we let  $n = m(2)$  and the vertices  $w_0, w_1, \dots, w_n$  (see condition (2) in Theorem 4.1) be given by  $w_0 = u_0, w_1 = a_1^2, \dots, w_{n-1} = a_{n-1}^2, w_n = v_0$ .

If not, we repeat the trisection, so on. It is clear that there is  $k \in \mathbb{N}$  such that after  $k$  trisections the condition  $d(a_0^k, a_{n(k)}^k) \geq \frac{1}{2}d(u_0, v_0)$  is satisfied. At that point we define  $n = n(k)$  and  $w_0, \dots, w_n$ , by  $w_0 = u_0, w_1 = a_1^k, \dots, w_{n-1} = a_{n-1}^k, w_n = v_0$ .

Now we introduce  $z_1, \dots, z_n$  and  $\tilde{z}_1, \dots, \tilde{z}_n$ . Observe that  $a_0^k, \dots, a_n^k$  are vertices in  $N_{j+k-1}$ . We let  $z_i$   $i = 1, \dots, n$  be the midpoint on  $S$  between  $a_{i-1}^k$  and  $a_i^k$ . Now we recall that when we create  $X_{j+k}$  we consider two copies of  $X_{j+k-1}$  and paste them at  $N_{j+k-1}$ . After that (in the notation introduced above) points  $z_i$  are identified with pairs  $(z_i, 0)$ . We introduce  $\tilde{z}_i$  as  $(z_i, 1)$ . It remains to verify that condition (15) is satisfied.

It is easy to see that  $d(z_i, \tilde{z}_i) = d(a_{i-1}^k, a_i^k)$ . Therefore

$$\sum_{i=1}^n d(z_i, \tilde{z}_i) = d(a_0^k, a_n^k) \geq \frac{1}{2}d(u_0, v_0). \quad \square$$

Combining Proposition 4.3 with Theorem 4.1 we get the following result of Cheeger and Kleiner [CK09, Corollary 1.7].

**Corollary 4.4.**  *$X_\omega$  does not admit a bilipschitz embedding into a Banach space with the RNP.*



### 4.3 A wider class of metric spaces which do not admit bilipschitz embeddings into RNP spaces

The purpose of this section is to prove an “isomorphic” version of Theorem 4.1. We need the following notion.

**Definition 4.5.** Let  $C \in [1, \infty)$  and  $u, v$  are two points in a metric space. A  $C$ -geodesic between  $u$  and  $v$ , also called a  $Cuv$ -geodesic, is a finite sequence

$$u_0 = u, u_1, u_2, \dots, u_n = v$$

of points satisfying

$$\sum_{i=1}^n d(u_i, u_{i-1}) \leq C \cdot d(u, v). \quad (23)$$

A  $Cuv$ -geodesic  $v_0, \dots, v_m$  is called an *extension* of a  $Cuv$ -geodesic  $u_0, \dots, u_n$  if  $m \geq n$  and  $u_0, \dots, u_n$  is a subsequence of  $v_0, \dots, v_m$ ,

*Remark 4.6.* This notion of a  $C$ -geodesic is in a certain sense too wide because the distances  $d(u_i, u_{i+1})$  are allowed to be “large”. In particular, a  $Cuv$ -geodesic can be trivial: the sequence  $u_0, u_1$  with  $u_0 = u$  and  $u_1 = v$  is a  $Cuv$ -geodesic for each  $C \geq 1$ .

We are going to construct martingales corresponding to bilipschitz embeddings of metric spaces having such geodesics. In this connection we introduce the following terminology.

**Definition 4.7.** Let  $u_0 = u, u_1, u_2, \dots, u_n = v$  be a  $Cuv$ -geodesic. We introduce the numbers

$$\alpha_k := \frac{\sum_{i=1}^k d(u_i, u_{i-1})}{\sum_{i=1}^n d(u_i, u_{i-1})}, k = 1, \dots, n, \quad \alpha_0 = 0. \quad (24)$$

These numbers induce a partition of the interval  $[0, 1]$  (for our purposes it does not matter how we include the points  $\{\alpha_i\}_{i=0}^n$  into the intervals of the partition). We call this partition the *partition corresponding to the  $Cuv$ -geodesic  $u_0, u_1, u_2, \dots, u_n$* .

If we consider a sequence of  $Cuv$ -geodesics in which each next  $Cuv$ -geodesic is an extension of the previous one, we form the following increasing sequence of partitions of  $[0, 1]$ . The first partition is the partition corresponding to the first  $Cuv$ -geodesic. The second partition is a refinement of the first partition obtained in the following way. Let  $\alpha_i$  and  $\alpha_{i+1}$  be two consecutive points in the first partition  $w_i$  and  $w_{i+1}$  be the corresponding points in the geodesic. Let  $z_j = w_i, \dots, z_k = w_{i+1}$  be the corresponding points in the extended  $Cuv$ -geodesic. The interval  $[\alpha_i, \alpha_{i+1}]$  in the refinement is partitioned by the points

$$\begin{aligned} \alpha_i, \alpha_i + \frac{(\alpha_{i+1} - \alpha_i)d(z_j, z_{j+1})}{\sum_{t=j}^{k-1} d(z_t, z_{t+1})}, \alpha_i + \frac{(\alpha_{i+1} - \alpha_i) \sum_{t=j}^{j+1} d(z_t, z_{t+1})}{\sum_{t=j}^{k-1} d(z_t, z_{t+1})}, \dots, \\ \alpha_i + \frac{(\alpha_{i+1} - \alpha_i) \sum_{t=j}^{k-2} d(z_t, z_{t+1})}{\sum_{t=j}^{k-1} d(z_t, z_{t+1})}, \alpha_{i+1}. \end{aligned}$$

The same is done for each interval of the partition. This procedure of refinement is repeated for each further extension of *Cuv*-geodesics.

**Theorem 4.8.** *Let  $(X, d)$  be a metric space for which there are two points  $u$  and  $v$  in  $X$  and a family of marked *Cuv*-geodesics such that the following conditions are satisfied:*

1. *Any marked *Cuv*-geodesic*

$$w_0 = u, w_1, \dots, w_{n-1}, w_n = v$$

*has two different marked extensions*

$$z_0 = u, z_1, \dots, z_m = v \tag{25}$$

*and*

$$\tilde{z}_0 = u, \tilde{z}_1, \dots, \tilde{z}_m = v. \tag{26}$$

*These extensions are such that for any sequence  $z'_0, z'_1, \dots, z'_m$  in which each  $z'_i$  is either  $z_i$  or  $\tilde{z}_i$  is also a marked *Cuv*-geodesic. Furthermore, the extensions (25) and (26) satisfy the following conditions:*

- (a) *They have some more common points in addition to  $w_0, w_1, \dots, w_{n-1}, w_n$ , and all common points  $\{z_i\}_{i \in C}$  have the same indices in both sequences, and form a marked *Cuv*-geodesic.*
- (b) *Have some pairs of distinct points  $\{z_i, \tilde{z}_i\}_{i \in D}$  which satisfy*
  - *Each pair of distinct points is between two pairs of common points.*

$$\frac{d(z_i, z_{i-1})}{d(z_i, z_{i+1})} = \frac{d(\tilde{z}_i, \tilde{z}_{i-1})}{d(\tilde{z}_i, \tilde{z}_{i+1})}. \tag{27}$$

$$\sum_{i \in D} d(z_i, \tilde{z}_i) \geq cd(u, v), \tag{28}$$

*where  $c$  does not depend on the choice of a marked *Cuv*-geodesic*

$$w_0 = u, w_1, \dots, w_{n-1}, w_n = v.$$

2. *The set of all marked *C*-geodesics satisfies the following condition of “non-accumulation of distortion of partitions”: there exists a constant  $B \in [1, \infty)$  such that if we consider a collection of marked *Cuv*-geodesics with each next being an extension of the previous one and consider the corresponding nested family of partitions, each next refining the previous one (according to the construction of Definition 4.7), then the final partition will be  $B$ -equivalent to the one corresponding to the last geodesic in the sequence. By  $B$ -equivalence we mean that*

$$\frac{1}{B} \leq \frac{\alpha_{i+1} - \alpha_i}{\beta_{i+1} - \beta_i} \leq B.$$

Then the metric space  $(X, d)$  does not admit bilipschitz embeddings into Banach spaces with the Radon-Nikodým property.

*Proof.* We assume that  $(X, d)$  admits a bilipschitz embedding  $f : X \rightarrow Y$  into a Banach space  $Y$  and show that there exists a bounded divergent martingale on  $[0, 1]$  with values in  $Y$ . We assume that

$$\ell d(x, y) \leq \|f(x) - f(y)\|_Y \leq d(x, y) \quad (29)$$

for some  $\ell > 0$ .

We are going to construct a bounded divergent  $Y$ -valued martingale. We start our construction of the martingale by picking any marked  $Cuv$ -geodesic  $w_0 = u, w_1, \dots, w_{n-1}, w_n = v$ . Let  $\alpha_0 = 0, \alpha_1, \dots, \alpha_n = 1$  be the ends of the corresponding partition of  $[0, 1]$ . We introduce the first function of the martingale,  $M_0 : [0, 1] \rightarrow Y$ , by

$$M_0(t) = \frac{f(w_{i+1}) - f(w_i)}{\alpha_{i+1} - \alpha_i} \quad \text{for } t \in [\alpha_i, \alpha_{i+1}]$$

All functions of our martingale will be of this type for partitions obtained by sequences of refinements done according to Definition 4.7. We observe that condition 2 of Theorem 4.8 in combination with (24) and (29) implies the boundedness of the martingale. In fact, in the case where  $[\alpha_i, \alpha_{i+1}]$  form a partition corresponding to a  $Cuv$ -geodesic  $w_0 = u, w_1, \dots, w_{n-1}, w_n = v$  we have

$$\left\| \frac{f(w_{i+1}) - f(w_i)}{\alpha_{i+1} - \alpha_i} \right\| = \left\| \frac{(f(w_{i+1}) - f(w_i)) \sum_{j=1}^n d(w_i, w_{j-1})}{d(w_i, w_{i+1})} \right\| \leq C d(u, v),$$

where we use (29) and the definition of a  $C$ -geodesic. In the further steps the boundedness of the integral follows from condition 2 of Theorem 4.8.

In the next step we apply condition 1 of Theorem 4.8 to the marked  $Cuv$ -geodesic  $w_0 = u, w_1, \dots, w_{n-1}, w_n = v$ . We get two marked  $Cuv$ -geodesics

$$z_0 = u, z_1, \dots, z_m = v$$

and

$$\tilde{z}_0 = u, \tilde{z}_1, \dots, \tilde{z}_m = v.$$

Let  $\{z_i\}_{i \in C} = \{\tilde{z}_i\}_{i \in C}$  be the set of their common points, which also form a marked  $Cuv$ -geodesic, we denote it  $\{y_i\}$ . We consider the refinement of the partition given by  $\alpha_i$  corresponding to the extension  $\{y_i\}$  of the geodesic  $\{w_i\}$ . Let  $\{\beta_i\}$  be the points of division of the corresponding partition of  $[0, 1]$ . Then we divide the subintervals containing distinct points in the corresponding proportion. The condition (27) implies that we get the equal divisions for both geodesics. We denote the obtained division points  $\{\gamma_i\}_{i=0}^m$ ,  $\gamma_0 = 0$ ,  $\gamma_m = 1$ .

Now we define two next functions of the martingale:

$$M_1(t) = \frac{f(y_{i+1}) - f(y_i)}{\beta_{i+1} - \beta_i} \quad \text{for } t \in [\beta_i, \beta_{i+1}].$$

It is clear that  $M_0$  is the conditional expectation of  $M_1$  with respect to the  $\sigma$ -algebra generated by  $\alpha$ .

In the next step we do the same for the points  $z_i$  and  $\tilde{z}_i$  and numbers  $\{\gamma_i\}$ . The only difference is that now for each  $i \in D$  we pick either  $z_i$  or  $\tilde{z}_i$ . We definitely get a martingale, no matter whether we pick  $z_i$  or  $\tilde{z}_i$  in each of the steps. The point of the choice is to make the (eventually constructed) martingale divergent. To achieve this goal in we observe that for each  $i \in D$  at least one of the following inequalities holds:

$$\left\| \frac{f(z_i) - f(z_{i-1})}{\gamma_i - \gamma_{i-1}} - \frac{f(z_{i+1}) - f(z_i)}{\gamma_{i+1} - \gamma_i} \right\| \geq \frac{\ell}{2} d(z_i, \tilde{z}_i) \left( \frac{1}{\gamma_i - \gamma_{i-1}} + \frac{1}{\gamma_{i+1} - \gamma_i} \right) \quad (30)$$

or

$$\left\| \frac{f(\tilde{z}_i) - f(z_{i-1})}{\gamma_i - \gamma_{i-1}} - \frac{f(z_{i+1}) - f(\tilde{z}_i)}{\gamma_{i+1} - \gamma_i} \right\| \geq \frac{\ell}{2} d(z_i, \tilde{z}_i) \left( \frac{1}{\gamma_i - \gamma_{i-1}} + \frac{1}{\gamma_{i+1} - \gamma_i} \right) \quad (31)$$

In fact, otherwise we get

$$\|f(z_i) - f(\tilde{z}_i)\| \left( \frac{1}{\gamma_i - \gamma_{i-1}} + \frac{1}{\gamma_{i+1} - \gamma_i} \right) < \ell d(z_i, \tilde{z}_i) \left( \frac{1}{\gamma_i - \gamma_{i-1}} + \frac{1}{\gamma_{i+1} - \gamma_i} \right),$$

we get a contradiction with (29). This choice of the *Cuv*-geodesic allows us to get a lower estimate for  $\int_0^1 \|M_2(t) - M_1(t)\| dt$ . We start by estimating the part of this difference corresponding to the interval  $[\gamma_{i-1}, \gamma_{i+1}]$ . Since the restriction of  $M_2$  to this interval is a two-valued function, and  $M_1$  is constant on the interval, integral

$$\int_{\gamma_{i-1}}^{\gamma_{i+1}} \|M_2 - M_1\| dt \quad (32)$$

can be estimated from below in the following way. Denote the value of  $M_2$  on the first part of the interval by  $x$ , the value on the second by  $y$ , the value of  $M_1$  on the whole interval by  $z$ , the length of the first interval by  $A_1$  and of the second by  $A_2$ . We have: the desired integral is equal to  $A_1\|x - z\| + A_2\|y - z\|$  and therefore can be estimated in the following way:

$$\begin{aligned} A_1\|x - z\| + A_2\|y - z\| &\geq \max\{\|x - z\|, \|y - z\|\} \cdot \min\{A_1, A_2\} \\ &\geq \frac{1}{2}\|x - y\| \min\{A_1, A_2\}. \end{aligned}$$

Therefore, assuming without loss of generality that the left-hand side of (30) is larger than the left-hand side of (31), the integral in (32) can be estimated from below by

$$\frac{1}{2} \cdot \frac{\ell}{2} d(z_i, \tilde{z}_i) \left( \frac{1}{\gamma_i - \gamma_{i-1}} + \frac{1}{\gamma_{i+1} - \gamma_i} \right) \min\{\gamma_i - \gamma_{i-1}, \gamma_{i+1} - \gamma_i\} \geq \frac{\ell}{4} d(z_i, \tilde{z}_i).$$

Summing over all intervals and using the condition (28), we get

$$||M_2 - M_1|| \geq \frac{\ell}{4} c d(u, v). \quad (33)$$

We continue in an obvious way. It is clear that similarly to (33) we get

$$||M_{2n} - M_{2n-1}|| \geq \frac{\ell}{4} c d(u, v)$$

for each  $n \in \mathbb{N}$ . □

#### 4.4 Metric spaces satisfying the conditions of Theorem 4.1 or 4.8 are not test spaces for the RNP

**Theorem 4.9.** *For each metric space  $X$  satisfying the conditions of Theorem 4.1 or 4.8 there exists a subspace of  $L_1(0, 1)$  which does not have the RNP and does not admit a bilipschitz embedding of  $X$ . Hence none of such metric spaces is a test space for the RNP.*

Proof of this theorem is based on the following result of Bourgain and Rosenthal [BR80] (see also expositions of this result in [BL00, Section 4 of Chapter 5] and [Bou83]).

**Theorem 4.10** (Bourgain-Rosenthal). *Let  $\{k_n\}_{n=1}^\infty$  be any increasing sequence of positive integers. There exists a subspace  $E$  of  $L_1(0, 1)$  which does not have the Radon-Nikodým property, but is such that each  $E$ -valued martingale  $\{f_n\}_{n=1}^\infty$  on  $[0, 1]$  adapted to a sequence  $\{\mathcal{B}_n\}_{n=1}^\infty$  of finite  $\sigma$ -algebras satisfying  $|\mathcal{B}_n| \leq k_n$  and bounded in the sense that  $\sup_n ||f_n||_\infty \leq 1$  satisfies  $\liminf_{n \rightarrow \infty} ||f_{n+1} - f_n||_1 = 0$ .*

This theorem is almost mentioned in [BR80, top of page 55] and [BL00, Remark on page 121]. However the statements are somewhat different. For this reason we describe the modifications which should be done in the proof of the main result of [BR80] in order to get it in the form of Theorem 4.10. (I decided not to reproduce the whole proof because this would lead to too much copying.)

We describe the modifications needed to get a proof of Theorem 4.10 out of the proof following the Example 5.30 in [BL00].

1. We replace  $2^{-N(m, \varepsilon)}$  in the definition of  $\delta(F, \varepsilon)$  by  $1/k_{[N(m, \varepsilon)]}$ , where  $k_{[N(m, \varepsilon)]}$  is the corresponding term of the sequence  $\{k_n\}_{n=1}^\infty$ .
2. We replace the assumption at the bottom of page 119 by the assumption that there is a  $E$ -valued martingale  $\{f_n\}$  on  $[0, 1]$  adapted to some sequence  $\{\mathcal{B}_n\}_{n=1}^\infty$  of finite  $\sigma$ -algebras satisfying  $|\mathcal{B}_n| \leq k_n$  and bounded in the sense that  $\sup_n ||f_n||_\infty \leq 1$
3. In the second paragraph on page 120 we replace the corresponding sentences by: The value of  $g_m(t) - f_m(t)$  is a convex combination of at most  $k_s$  values

of  $g_s - f_s$  on the  $\mathcal{B}_m$ -atom containing  $t$ . Using the estimate  $\mu\{|h| \geq \delta\} \leq k \max \mu\{|h_i| \geq \delta\}$  whenever  $h$  is a *convex combination* of the  $k$  functions  $\{h_i\}$ , we see that for every  $0 \leq t \leq 1$

$$d(g_m(t), f_m(t)) \leq k_s \delta(F_n, \varepsilon_n) \leq \gamma(F_n, \varepsilon_n).$$

The rest of the proof is the same.

*Proof of Theorem 4.9.* Theorem 4.10 is not immediately applicable to the martingales which we construct in Theorems 4.1 and 4.8. There are two problems with its applicability:

- (1) In our construction we can claim the norms  $\|M_{2n} - M_{2n-1}\|$  are bounded away from 0, but there are no lower bounds on  $\|M_{2n+1} - M_{2n}\|$ .
- (2) Another problem is that the sequence of finite *sigma*-algebras  $\{\mathcal{B}_n\}_{n=0}^\infty$  which we get in our construction depends not only on the metric space which we consider, but also on the embedding and the Banach space into which we embed.

We can easily find a way around obstacle (1). We just consider the martingale  $\{M_{2n-1}\}_{n=1}^\infty$ . Since the conditional expectation is a contraction on  $L_1(Y)$  even in the Banach-space-valued case (see [Pis11, Section 1.1]), applying the conditional expectation with respect to  $\mathcal{B}_{2n}$  to  $M_{2n+1} - M_{2n-1}$  we get

$$\|M_{2n+1} - M_{2n-1}\|_1 \geq \|M_{2n} - M_{2n-1}\|_1.$$

Therefore the norms of all differences of the martingale  $\{M_{2n-1}\}_{n=1}^\infty$  are bounded away from 0 in  $L_1(Y)$ .

Now we analyze obstacle (2). Let  $X$  be any metric space satisfying the conditions of Theorem 4.1 or 4.8. By the corresponding proof this implies that for any bilipschitz map  $f : X \rightarrow Y$  into a Banach space  $Y$  we can construct a bounded  $Y$ -valued martingale  $\{M_n\}$  such that  $\inf_n \|M_{2n} - M_{2n-1}\| \geq \delta > 0$ . By the previous remark we can create out of it a new martingale  $S_k$  with  $\inf_k \|S_k - S_{k-1}\| \geq \delta > 0$ . Now about the corresponding  $\sigma$ -algebras. The  $\sigma$ -algebras corresponding to  $M_0$ ,  $M_1$  and  $M_2$  depend only of the metric space  $X$  and the choice of points which we make in the first step. The  $\sigma$ -algebras  $\mathcal{B}_3$  and  $\mathcal{B}_4$  depend also on the choices of  $z_i$  and  $\tilde{z}_i$  which we make in the first step of the construction. Therefore there are many different choices for the algebras  $\mathcal{B}_3$  and  $\mathcal{B}_4$ , the choice that we have to make depends on the space  $Y$  and the embedding  $f$ . Important point is that there are finitely many options (because there are finitely many ways to choose different collections  $\{z'_i\}$ ). Thus we have shown the following claim.

**Claim 4.11.** *For each  $n \in \mathbb{N}$  there exist finitely many finite  $\sigma$ -algebras  $\{\mathcal{B}_{n,j}\}_{j=1}^{m(n)}$  such that for each Banach space  $Y$  and each bilipschitz embedding of  $M$  into  $Y$  there is a bounded divergent martingale  $\{S_n\}_{n=1}^\infty$  with respect to a filtration  $\{\mathcal{B}_n\}_{n=1}^\infty$ , where each  $\mathcal{B}_n$  is one of  $\{\mathcal{B}_{n,j}\}_{j=1}^{m(n)}$ , and such that  $\sup_n \|S_n\|_\infty < \infty$  and  $\inf_n \|S_{n+1} - S_n\|_1 \geq \delta > 0$ .*

It remains to let

$$k_n = \max_{j \in \{1, \dots, m(n)\}} |\mathcal{B}_{n,j}|$$

and to apply Theorem 4.10.  $\square$

## 5 A submetric test space for reflexivity

The purpose of this section is to show that the well-known (linear) characterization of reflexivity leads to a submetric test space characterization of reflexivity.

Let  $\Delta \geq 1$ . The submetric space  $X_\Delta$  is the space  $\ell_1$  with its usual metric. The only thing which makes it different from  $\ell_1$  is the set of active pairs  $S_\Delta$ : A pair  $(x, y) \in X_\Delta \times X_\Delta$  is active if and only if

$$\|x - y\|_1 \leq \Delta \|x - y\|_s, \quad (34)$$

where  $\|\cdot\|_s$  is the summing norm, that is,

$$\|\{a_i\}_{i=1}^\infty\|_s = \sup_k \left| \sum_{i=1}^k a_i \right|.$$

**Theorem 5.1.**  *$X_\Delta$ ,  $\Delta \geq 2$  is a submetric test space for reflexivity.*

We start by recalling the characterization of reflexivity developed in a series of papers around 1960: [Pta59, Sin62, Pel62, Jam64b, MM65]. We state it in the following way (we use the standard terminology of [LT77]):

**Theorem 5.2.** *For each  $0 < \theta < 1$  there exists  $1 < B < \infty$  such that a Banach space  $Y$  is non-reflexive if and only if there is a basic sequence  $\{y_i\}_{i=1}^\infty \subset Y$  with basic constant  $B$  and  $\|y_i\| = 1$ ; and a functional  $f \in Y^*$  such that  $\|f\| = 1$  and  $f(y_i) = \theta$  for all  $i \in \mathbb{N}$ .*

*Proof of Theorem 5.1.* Suppose that  $Y$  is nonreflexive and show that in such a case  $\ell_1$  admits a *partially* bilipschitz embedding with the set of active pairs  $S_\Delta$ . Let  $\{y_i\} \subset Y$  and  $f \in Y^*$  satisfy conditions of Theorem 5.2. We embed  $X_\Delta$  into  $Y$  in the following way: we map a unit vector  $e_i$  of  $\ell_1$  to  $y_i$ , and extend this map by linearity. It is clearly a Lipschitz map (on the whole space  $\ell_1$ ).

We need to estimate  $\|u - v\|_X$  from below in terms of  $\|u - v\|_1$  for an active pair  $(u, v)$ . We let  $u = \sum_i u_i y_i$ ,  $v = \sum_i v_i y_i$ . We have

$$\begin{aligned} \|u - v\|_X &\geq \sup_k \frac{\|\sum_{i=1}^k (u_i - v_i) y_i\|}{B} \geq \sup_k \frac{|f(\sum_{i=1}^k (u_i - v_i) y_i)|}{B} \\ &= \theta \sup_k \frac{|\sum_{i=1}^k (u_i - v_i)|}{B} = \frac{\theta}{B} \|u - v\|_s \geq \frac{\theta}{B\Delta} \|u - v\|_1. \end{aligned}$$



Now let  $T : \ell_1 \rightarrow Y$  be a partially bilipschitz embedding with the set of active pairs  $S_\Delta$  and constant  $C$  (see (1)). Observe that each vector in  $z \in \ell_1$  can be represented as a difference of two vectors  $z = x_1 - x_2$  with positive coordinates, for which  $\|x_1\|_1 = \|x_1\|_s$ ,  $\|x_2\|_1 = \|x_2\|_s$  and  $\|z\|_1 = \|x_1\|_1 + \|x_2\|_1$ . Therefore a partially bilipschitz embedding of  $X_\Delta$  into  $Y$  is a Lipschitz map of  $\ell_1$  into  $Y$ :

$$\begin{aligned} \|T(y+z) - T(y)\|_Y &\leq \|T(y+z) - T(y+x_1)\|_Y + \|T(y+x_1) - Ty\|_Y \\ &\leq C\|x_2\|_1 + C\|x_1\|_1 \\ &= C\|z\|_1. \end{aligned} \tag{35}$$

If  $Y$  does not have the RNP, then  $Y$  is nonreflexive (as is well known), see [BL00], and there is nothing to prove in this case. If  $Y$  has the RNP, then, by the theorem of Aronszajn [Aro76], Christensen [Chr73], and Mankiewicz [Man73] (see [BL00, Theorem 6.42]), there is a point  $p$  of Gâteaux differentiability of  $T$  in  $\ell_1$ . Let  $D$  be the Gâteaux derivative of  $T$  at  $p$ . Since  $T$  is partially bilipschitz, for each  $z \in \ell_1$  satisfying  $\|z\|_1 \leq \Delta\|z\|_s$  we have

$$\|z\|_1 \leq \|T(p+z) - T(p)\|_Y \leq C\|z\|_1.$$

Using this inequality, (35), and the definition of the Gâteaux derivative we get that  $D : \ell_1 \rightarrow Y$  is a linear operator satisfying  $\|D\| \leq C$  and

$$\|Dz\|_Y \geq \|z\|_1 \text{ for each } z \in \ell_1 \text{ satisfying } \|z\|_1 \leq \Delta\|z\|_s. \tag{36}$$

Let  $u_i = De_i$ , where  $\{e_i\}$  is the unit vector basis of  $\ell_1$ . Then the sequence  $\{u_i\}$  satisfies the conditions:

1.  $1 \leq \|u_i\| \leq C$ .
2.  $\forall i \in \mathbb{N} \quad \text{dist}(\text{conv}(u_1, \dots, u_i), \text{conv}(u_{i+1}, \dots)) \geq 2$ .

In fact, the first condition follows from  $\|D\| \leq C$ . The second condition follows from the inequality

$$\left\| \sum_{j=1}^i \alpha_j e_j - \sum_{j=i+1}^{\infty} \alpha_j e_j \right\|_1 \leq 2 \left\| \sum_{j=1}^i \alpha_j e_j - \sum_{j=i+1}^{\infty} \alpha_j e_j \right\|_s,$$

satisfied whenever  $\alpha_j$  are nonnegative and satisfy  $\sum_{j=1}^i \alpha_j = 1$  and  $\sum_{j=i+1}^{\infty} \alpha_j = 1$ . Therefore (recall that  $\Delta \geq 2$ )

$$\begin{aligned} \left\| D \left( \sum_{j=1}^i \alpha_j e_j - \sum_{j=i+1}^{\infty} \alpha_j e_j \right) \right\|_Y &= \left\| \sum_{j=1}^i \alpha_j u_j - \sum_{j=i+1}^{\infty} \alpha_j u_j \right\|_Y \\ &\stackrel{(36)}{\geq} \left\| \sum_{j=1}^i \alpha_j e_j - \sum_{j=i+1}^{\infty} \alpha_j e_j \right\|_1 = 2. \end{aligned}$$

It is well known (and is a version of the characterization of reflexivity developed in [Pta59, Sin62, Pel62, Jam64b, MM65]; see [Bea82, pp. 49–55] and [Ost13b, Chapter 6]) that the existence of such sequence  $\{u_j\}$  in  $Y$  implies nonreflexivity of  $Y$ .  $\square$

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